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Interaction Theory: Relativistic Hydrogen Atom and the Lamb Shift

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The Dirac relativistic equation for the hydrogen atom as augmented by dual interaction terms is solved exactly and the eigenvalues for the bound states are determined. These are $E_n = \mu[1 + \kappa^2/(n + s)^2]^{-1/2}$, with $s^2 = k^2 - \kappa^2(1 \pm \Lambda)^2$, where Λ is a constant which is a measure of the strength of the dual interaction relative to the standard interaction $\gamma^\mu A_\mu$. It is shown that the ratios of the experimental values for the Lamb shifts of various energy levels in hydrogen and singly ionized helium are correctly given by the theory. The origin of the anomalous magnetic moment and, in fact, the operator for the total magnetic moment is given.

1. INTRODUCTION

The work to be presented is, in a real sense, a test of *interaction theory*, what may be called the *experimentum crucis* for the theory. Therefore, it is fortunate that the problem that is undertaken can be solved exactly. The relativistic hydrogen atom as treated by Dirac will be altered to the extent required by interaction theory (Schwebel, 1972). From an analysis of electromagnetic theory, we found that each conventional interaction quantity, such as force, energy, etc., has a dual in the tensor sense. It is these additional quantities that have been added to the well-known Dirac equation to form interaction theory's version of the relativistic hydrogen atom.

We will solve this new problem and obtain the exact eigenvalues for the bound states. Once these have been obtained we will find that the Lamb shift is, up to an experimentally determined constant, properly accounted for, in fact, determined for all energy levels. Moreover, we will show that the experimentally determined ratios of the Lamb shifts for known energy levels of hydrogen are, to good approximation, the same as those for singly

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ionized helium atoms. And this is, to the same approximations, the theoretical result.

The following section will contain the augmented Dirac equation for the relativistic hydrogen atom and its exact solution. The next section contains the computation of the Lamb shifts and their comparison with experimental data. The last section will critically examine the results obtained, their limitations, and the problems that arise which the theory must solve. In particular, the operator for the magnetic moment of the electron will be given. It contains a contribution from dual interactions which is responsible for the "anomalous" behavior of the magnetic moment.

2. RELATIVISTIC EQUATION FOR THE HYDROGEN ATOM

Dirac's equation for the hydrogen atom as augmented by terms from the interaction theory of the electromagnetic field is

$$[-i\gamma^\mu \partial_\mu + \gamma^\mu(1 + \lambda\gamma^5)A_\mu(q)e^p + \mu]\psi = 0 \tag{2.1}$$

Much of the notation is standard, except for λ, which is a constant to be determined experimentally. It is a measure of the strength of the dual contribution relative to the conventional term. The four-potential $A_\mu(q)$ is due to the proton, labeled the qth particle, and $e^p < 0$ is the electronic charge. We have chosen units such that $\hbar = c = 1$. The mass μ is the reduced mass of the system. We also have

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$(\gamma^0)^2 = 1, \qquad (\gamma^k)^2 = (\gamma^5)^2 = -1, \qquad k = 1, 2, 3$$

The σ's are the Pauli matrices.

In spherical coordinates equation (2.1) for the hydrogen atom becomes

$$\left[\gamma_r p_r + i\gamma_r\gamma^0 \frac{K}{r} + \gamma^0(1 + \lambda\gamma^5)\frac{\kappa}{r} - \gamma^0 E + \mu\right]\psi = 0 \tag{2.2}$$

in which we have chosen the standard time dependence for the wave function and

$$\gamma_r = \boldsymbol{\gamma}\cdot\hat{\mathbf{r}}, \qquad p_r = \hat{\mathbf{r}}\cdot\mathbf{p} - \frac{i}{r}, \qquad \kappa = e^p e^q, \qquad \hat{r} = \frac{\mathbf{r}}{r}$$

$$K = -i\gamma^0(\gamma^0\gamma^5\boldsymbol{\gamma}\cdot\mathbf{L} + i) = \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{L} + 1 & 0 \\ 0 & -(\boldsymbol{\sigma}\cdot\mathbf{L} + 1) \end{pmatrix} \tag{2.3}$$

All operators satisfy the same commutation relations as in the Dirac case, except that K does not commute with the Hamiltonian. This is due to the dual interaction term, which contains γ^5 and

$$K\gamma^5 = -\gamma^5 K \tag{2.4}$$

However, K^2 does commute with γ^5 and the Hamiltonian. It follows from the requirement that the Hamiltonian be Hermitian that the constant λ must be imaginary, i.e., $\lambda = i\Lambda$ with Λ real.

Because of equation (2.4), the wave function is assumed to be a linear combination of the eigenfunctions of K with eigenvalues k and $-k$ ($k > 0$). The eigenvalues of $K^2 = (j + 1/2)^2$, are the familiar ones where j is half an odd integer. It follows that $k = j + 1/2$ takes on all positive integral values 1, 2, 3, ...

We introduce the notations

$$K\psi_+ = k\psi_+, \qquad K\psi_- = -k\psi_-$$

and

$$\psi_+ = \begin{pmatrix} f_1(r)\psi^u \\ g_1(r)\psi^l \end{pmatrix}$$

In the latter, $f_1(r)$ and $g_1(r)$ are scalar functions of r and ψ^u, ψ^l are two component spinors which are functions of the angles and satisfy well-known relations (Corinaldesi, 1963),

$$(\boldsymbol{\sigma}\cdot\mathbf{L} + 1)\psi^u = k\psi^u$$

$$(\boldsymbol{\sigma}\cdot\mathbf{L} + 1)\psi^l = -k\psi^l \tag{2.5}$$

and

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\psi^u = \psi^l$$

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\psi^l = \psi^u \tag{2.6}$$

It follows from equation (2.3) and (2.5) that

$$\psi_- = \begin{pmatrix} f_2(r)\psi^l \\ g_2(r)\psi^u \end{pmatrix}$$

satisfies

$$K\psi_- = -k\psi_-$$

To reduce equation (2.2) to one dependent only on r, we assume that the wave function has the form

$$\psi = \psi_+ + \psi_-$$

Substituting it into equation (2.2), using equations (2.3), (2.5), (2.6), and the orthogonality of ψ^l and ψ^u, we obtain the following set of equations:

$$\begin{aligned}
-i\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)g_1-\frac{ik}{r}g_1+\frac{\kappa}{r}f_1-\frac{i\kappa\lambda}{r}g_2+(\mu-E)f_1=0\\
-i\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)g_2+\frac{ik}{r}g_2+\frac{\kappa}{r}f_2-\frac{i\kappa\lambda}{r}g_1+(\mu-E)f_2=0\\
+i\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)f_1-\frac{ik}{r}f_1-\frac{\kappa}{r}g_1+\frac{i\kappa\lambda}{r}f_2+(\mu+E)g_1=0\\
i\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)f_2+\frac{ik}{r}f_2-\frac{\kappa}{r}g_2+\frac{i\kappa\lambda}{r}f_1+(\mu+E)g_2=0
\end{aligned} \tag{2.7}$$

Next we set

$$f_1=F_1/r,\qquad f_2=iF_2/r,\qquad g_1=iG_1/r,\qquad g_2=G_2/r,\qquad \text{and}\qquad \lambda=i\Lambda$$

and write the resulting equations as a matrix

$$\left(\frac{d}{dr}+\frac{M}{r}+A\right)\phi=0$$

with

$$M=\begin{pmatrix} k & 0 & \kappa\Lambda & \kappa\\ 0 & k & \kappa & \kappa\Lambda\\ -\kappa\Lambda & -\kappa & -k & 0\\ -\kappa & -\kappa\Lambda & 0 & -k \end{pmatrix},\qquad \phi=\begin{pmatrix} G_1\\ F_2\\ G_2\\ F_1 \end{pmatrix} \tag{2.8}$$

and

$$A=\begin{pmatrix} 0 & 0 & 0 & \mu-E\\ 0 & 0 & -(\mu+E) & 0\\ 0 & E-\mu & 0 & 0\\ \mu+E & 0 & 0 & 0 \end{pmatrix}$$

To solve equation (2.7), we assume that

$$\phi=e^{-\alpha r}r^{n+s}|A_n\rangle \tag{2.9}$$

where summation over the repeated index n is over all positive integers and zero. Each $|A_n\rangle$ is a constant four-component column matrix and $|A_0\rangle\neq 0$. The constant s remains to be determined from the matrix counterpart to the indicial equation. The role of α is the usual one.

The substitution of equation (2.9) into equation (2.8) and the equating of coefficients of like powers of r leads to the relation

$$(n + s + M)|A_n\rangle + (A - \alpha)|A_{n-1}\rangle = 0 \tag{2.10}$$

For $n = 0$ and $|A_0\rangle \neq 0$ it follows that the determinant

$$|s + M| = 0$$

which requires that s be chosen so that

$$[s^2 - k^2 + \kappa^2(1 - \Lambda)^2][s^2 - k^2 + \kappa^2(1 + \Lambda)^2] = 0 \tag{2.11}$$

The solution, equation (2.9), in order to behave properly at the origin, $r = 0$, requires that $s > 0$, i.e.,

$$s = +[k^2 - \kappa^2(1 \pm \Lambda)^2]^{1/2} \tag{2.12}$$

For $n \geqslant 1$, equation (2.10) becomes

$$(n + s + M)|A_n\rangle = -(A - \alpha)|A_{n-1}\rangle \tag{2.13}$$

We solve this equation by first diagonalizing the matrix $(A - \alpha)$:

$$\begin{pmatrix} n + s + \frac{\kappa E}{\alpha} & -\frac{\kappa\Lambda(\mu + E)}{\alpha} & k - \frac{\kappa\mu}{\alpha} & 0 \\ \kappa\Lambda\frac{(\mu - E)}{\alpha} & n + s + \frac{\kappa E}{\alpha} & 0 & k + \frac{\kappa\mu}{\alpha} \\ k + \frac{\kappa\mu}{\alpha} & 0 & n + s - \frac{\kappa E}{\alpha} & \frac{\kappa\Lambda(\mu + E)}{\alpha} \\ 0 & k - \frac{\kappa\mu}{\alpha} & -\kappa\Lambda\frac{(\mu - E)}{\alpha} & n + s - \frac{\kappa E}{\alpha} \end{pmatrix} |A'_n\rangle$$

$$= 2\alpha \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} |A'_{n-1}\rangle \tag{2.14}$$

where we have chosen $\alpha = (\mu^2 - E^2)^{1/2}$ and the $|A'_n\rangle$ are the transformed $|A_n\rangle$.

Equation (2.14) is rewritten in terms of the components of $|A'_n\rangle$. We designate the first two components of $|A'_n\rangle$ by $|a_n\rangle$ and the last two by $|b_n\rangle$:

$$\begin{pmatrix} n + s + \frac{\kappa E}{\alpha} & -\frac{\kappa\Lambda(\mu + E)}{\alpha} \\ \kappa\Lambda\frac{(\mu - E)}{\alpha} & n + s + \frac{\kappa E}{\alpha} \end{pmatrix} |a_n\rangle + \begin{pmatrix} k - \frac{\kappa\mu}{\alpha} & 0 \\ 0 & k + \frac{\kappa\mu}{\alpha} \end{pmatrix} |b_n\rangle = 0$$

$$\begin{pmatrix} k + \frac{\kappa\mu}{\alpha} & 0 \\ 0 & k - \frac{\kappa\mu}{\alpha} \end{pmatrix} |a_n\rangle + \begin{pmatrix} n + s - \frac{\kappa E}{\alpha} & \kappa\Lambda \frac{(\mu + E)}{\alpha} \\ -\kappa\Lambda \frac{(\mu - E)}{\alpha} & n + s - \frac{\kappa E}{\alpha} \end{pmatrix} |b_n\rangle = 2\alpha |b_{n-1}\rangle$$

For large n, these equations reduce to

$$\left(n + s + \frac{\kappa E}{\alpha}\right)|a_n\rangle + \begin{pmatrix} k - \frac{\kappa\mu}{\alpha} & 0 \\ 0 & k + \frac{\kappa\mu}{\alpha} \end{pmatrix} |b_n\rangle = 0$$

$$\begin{pmatrix} k + \frac{\kappa\mu}{\alpha} & 0 \\ 0 & k - \frac{\kappa\mu}{\alpha} \end{pmatrix} |a_n\rangle + \left(n + s - \frac{\kappa E}{\alpha}\right)|b_n\rangle = 2\alpha |b_{n-1}\rangle$$

Eliminating $|b_n\rangle$, we get

$$\left[k^2 - \frac{\kappa^2\mu^2}{\alpha^2} - \left(n + s - \frac{\kappa E}{\alpha}\right)\left(n + s + \frac{\kappa E}{\alpha}\right)\right]|a_n\rangle = -2\alpha\left(n + s - 1 + \frac{\kappa E}{\alpha}\right)|a_{n-1}\rangle$$

As n increases we see that

$$|a_n\rangle \sim 2\alpha \frac{|a_{n-1}\rangle}{n}, \tag{2.15}$$

which is the same behavior of the coefficients as in the Dirac case (Schiff, 1949). As in that situation, the series does not converge sufficiently strongly to give a square integrable solution. Therefore, we require that

$$n + s + \frac{\kappa E_n}{\alpha} = 0 \tag{2.16}$$

where we have chosen E_n such that equation (2.16) is satisfied and, in order to simplify notation, consider n replaced by $n + 1$ in equation (2.15). Under these conditions we have $|a_{n+1}\rangle \sim 2\alpha|a_n\rangle/n^2$ and the convergence of the sequence is then sufficiently strong.

It follows from equation (2.16) that

$$E_n = \mu\left[1 + \frac{\kappa^2}{(n + s)^2}\right]^{-1/2} \tag{2.17}$$

where

$$s = [k^2 - \kappa^2(1 \pm \Lambda)^2]^{1/2}$$

In form, equation (2.17) is precisely that given in Dirac's solution to the hydrogen atom. If Λ is set equal to zero, then equation (2.17) is the same as Dirac's. However, if Λ is not zero, then there is not only the difference in the corresponding s values, but there is also the distinction that in s, k is always positive, whereas in Dirac's solution k may have both positive and negative values. This leads to the degeneracy of the levels, but in the above result, since k can only be positive, there are two levels with the same k but slightly different energy values. In other words, the degeneracy has been lifted.

3. LAMB SHIFT

Equation (2.17) for fixed n and k gives the exact energy values for the levels that are close in energy value and that would be identical in value were it not for the effect of the dual interaction. Hence the energy difference between two such levels is the Lamb shift.

We expand equation (2.17), with s as given, as a power series in the variable $\kappa^2(1 \pm \Lambda)^2$:

$$E_{n,k} = \mu\left[1 + \frac{\kappa^2}{(n+k)^2}\right]^{-1/2} - \frac{\mu}{2}\left[1 + \frac{\kappa^2}{(n+k)^2}\right]^{-3/2} \frac{\kappa^4(1 \pm \Lambda)^2}{k(n+k)^3} + \cdots \tag{3.1}$$

Fixing the values for n and k, $n' = n + k$, the corresponding Lamb shift is

$$\Delta E_{n,k} \equiv E_{n,k}(-\Lambda) - E_{n,k}(\Lambda) = \mu\left[1 + \frac{\kappa^2}{(n+k)^2}\right]^{-3/2} \frac{2\Lambda\kappa^4}{k(n+k)^3} + \cdots \tag{3.2}$$

up to the order $\kappa^2(1 \pm \Lambda)^2$.

The constant Λ, which is a measure of the weight of the dual interaction relative to the conventional interaction term, can be evaluated by equating the theoretical calculation to the experimentally determined value for the Lamb shift. This will be done, but it is interesting to consider the ratios of Lamb shifts for different values of n and k for the same atom. Such ratios, to the approximation given above, should be independent of Λ, and therefore of those characteristics of the atom upon which Λ may depend. Thus, we can consider hydrogenlike atoms, i.e., those that have been ionized to the extent that the remaining system consists of a nucleus about which we have an orbiting electron. Fortunately, there are data both for hydrogen and singly ionized helium atoms (Lautrup et al., 1972).

The data for hydrogen are ($n' = n + k$)

$$\begin{aligned}
\text{H } (n' = 2) \quad & \Delta E_{1,1} = 1057.90 \pm 0.06 \text{ MHz} \\
& \qquad\quad 1057.86 \pm 0.06 \text{ MHz} \\
\text{H } (n' = 3) \quad & \Delta E_{2,1} = 314.810 \pm 0.052 \text{ MHz} \\
\text{H } (n' = 4) \quad & \Delta E_{3,1} = 133.18 \pm 0.59 \text{ MHz}
\end{aligned}$$

$$\Delta E_{2,1}/\Delta E_{1,1} = 0.298 \qquad \Delta E_{3,1}/\Delta E_{1,1} = 0.126$$

and for singly ionized helium,

$$\begin{aligned}
\text{He}^+ \ (n' = 2) \quad & \Delta E_{1,1} = 14045.4 \pm 1.2 \text{ MHz} \\
\text{He}^+ \ (n' = 3) \quad & \Delta E_{2,1} = 4183.17 \pm 0.54 \text{ MHz} \\
\text{He}^+ \ (n' = 4) \quad & \Delta E_{3,1} = 1776.0 \pm 7.5 \text{ MHz} \\
& \qquad\quad 1768.0 \pm 5 \text{ MHz} \\
& \qquad\quad 1769.4 \pm 1.2 \text{ MHz}
\end{aligned}$$

$$\Delta E_{2,1}/\Delta E_{1,1} = 0.298 \qquad \Delta E_{3,1}/\Delta E_{1,1} = 0.126$$

So that within the accuracy of the data the ratios of the corresponding Lamb shifts are the same, which confirms, to the approximation made, the prediction of the theory.

The theoretical values for the same ratios as above are, from equation (3.2),

$$\frac{\Delta E_{2,1}}{\Delta E_{1,1}} = \left[\frac{4 + \kappa^2}{9 + \kappa^2}\right]^{3/2} \sim \frac{8}{27} = 0.296$$

$$\frac{\Delta E_{3,1}}{\Delta E_{1,1}} = \left[\frac{4 + \kappa^2}{16 + \kappa^2}\right]^{3/2} \sim \frac{1}{8} = 0.125$$

in which $\kappa^2 = (Z\alpha)^2$ has been neglected.

The results for hydrogen have been obtained earlier (Sachs and Schwebel, 1961). Those for ionized helium and for hydrogenlike atoms are the results of the present theory. The experimental data available seem to be adequately covered by the approximation given by equation (3.2) to the exact result, which is obtainable from equation (2.17). The latter equation is capable of giving the Lamb shifts for all the other energy levels and a test of its validity over these must await more experimental data.

The determination of Λ to the order of approximation given by equation (3.2) is, for hydrogen,

$$\Lambda_{\text{H}} = 1.2076 \times 10^{-2}$$

and for singly ionized helium

$$\Lambda_{He^+} = 1.0022 \times 10^{-2}$$

4. DISCUSSION

The difference in the value of Λ_H and Λ_{He^+} indicates that the Λ's must depend on physical parameters that distinguish the chemical elements. In particular the interaction between the magnetic moments of the nucleus and orbiting electron has been omitted. It is in that direction where we may hope to obtain some insight into the dependence of Λ on various physical parameters. A further indication that it is the magnetic moment interaction that must be investigated comes from the realization that the dual interaction results in a Lamb shift for the ground state of the hydrogen atom.

The energy shift is eight times as large as that for the level with $n' = 2$. But in addition to the Lamb shift of the ground state energy level, there is a hyperfine structure splitting due to the interaction between the magnetic moments of nucleus and orbiting electron. The resulting shift must be the net effect of the two interactions.

A study that takes into account both the magnetic moment interaction and the dual interaction that accompanies it is underway. The total magnetic moment operator (the conventional and its dual) has been found to be represented by the expression

$$\mu = \frac{e^p h}{2m^p c} i(1 + i\Lambda\gamma^5)\gamma^0\gamma^5\boldsymbol{\gamma}$$

It is interesting to note the addition of an "anomalous" term to the usual conventional magnetic moment operator. Moreover, the "anomalous" contribution to the magnetic moment operator is related to the constant Λ which appeared in the study of the Lamb shift. The work on this aspect of the problem is being prepared for publication.

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